University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION II

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Show all your work. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (25 points)
An idealization of the *Bainbridge mass spectrograph* consists of the following arrangement. The entire apparatus is immersed in a uniform constant magnetic field $\vec{B} = |B_0|\hat{z}$. In the semi-infinite volume $x < 0$, there is also a uniform electric field $\vec{E} = |E_0|\hat{y}$. At the point $(x = -|x_0|, y = 0, z = 0)$, a beam of nonrelativistic ions of positive charge q exists; there the ions have velocities that point exactly in the $+\hat{x}$ direction. At $x = 0$, a thin slit admits only those ions which still have $y = 0$. Thereafter, in the semi-infinite volume $x > 0$, the ions move under the influence only of the uniform magnetic field.

An ion detector is placed at $(x = 0, y = -|y_0|)$. If an ion is detected, what mass M does it have?
Solution:
In the region $-|x_0| < x < 0$, for the ion to remain at $y = 0$ in order to pass through the slit, the y component of the Lorentz force on it must vanish:

$$0 = F_y = q\big(E_y + (\vec{v} \times \vec{B})_y\big)$$
$$= q(E_y - v_x B_z)$$
$$v_x = \frac{|E_0|}{|B_0|} .$$

(This is a *crossed field velocity selector*.) In the region $x > 0$, the ion moves along the circumference of a circle of radius R, *clockwise* about $\hat{z}$. Since the magnetic field alone does no work, the ion's $|\text{velocity}|$ v remains equal to its initial velocity v_x. The ion circulates with angular

frequency equal to the cyclotron frequency ω_c:

$$\omega_c = \frac{v}{R} = \frac{q|B_0|}{M}$$
$$R = \frac{M v_x}{q|B_0|}$$
$$= \frac{M|E_0|}{qB_0^2} .$$

If the ion's mass is such that $2R = |y_0|$, it will complete a semicircle and intercept the detector at $(x = 0, y = -|y_0|)$. Therefore

$$\tfrac{1}{2}|y_0| = R = \frac{M|E_0|}{qB_0^2}$$
$$M = \frac{qB_0^2|y_0|}{2|E_0|} .$$

Problem 2. (35 points)
A cylindrical rod of radius b and length $L \gg b$ is centered on the origin and coaxial with the z axis. It has "frozen-in" magnetization $\vec{M} = |M_0|\hat{z}$.

(a) (15 points) To lowest order in b/L, what is the magnetic flux Φ_B through the rod in the $z = 0$ plane?
Solution:
On the curved surface of the rod, the bound surface current is

$$\vec{K}_b = \vec{M} \times \vec{n}$$
$$= \hat{\phi}|M_0| .$$

The magnetic field produced by this surface current is the same as that of a long solenoid. We apply Ampère's law over a rectangular loop $s = 0$

or $s = S > b$, $z = 0$ or $z = D$. Since the field outside a long solenoid is negligible compared to the field inside, only the leg $s = 0$ contributes to the integral $\oint \vec{B} \cdot d\vec{l}$. The field $B_0 \hat{z}$ inside the long solenoid is uniform and equal in magnitude to

$$B_0 D = \mu_0 K_B D$$
$$B_0 = \mu_0 |M_0| .$$

Therefore the flux through the area bounded by the circle $s = b$ is

$$\Phi_B = \pi b^2 \mu_0 |M_0| .$$

(b) (20 points) The flux Φ_B is returned through free space. Draw a circle centered on the origin in the $z = 0$ plane, of radius $s \gg L$ chosen so that 99% of the returned flux passes through the area bounded by the circle. To lowest order in b/L and L/s, what is s?

Solution:
Since $s \gg L$ (and $s \gg a$), the magnetic field at cylindrical radius s can be approximated by that of an ideal magnetic dipole with magnetic dipole moment

$$\vec{m} = \vec{M} V$$
$$= \hat{z} |M_0| \pi b^2 L ,$$

where V is the volume of the rod. At radius $\vec{r} = \hat{s} s$, the vector potential is

$$\frac{4\pi}{\mu_0} \vec{A} = \frac{\vec{m} \times \hat{r}}{r^2}$$
$$= \frac{m}{r^2} \hat{z} \times \hat{s}$$
$$= \hat{\phi} \frac{|M_0| \pi b^2 L}{s^2} .$$

The flux through the circular area within this radius is

$$\Phi_s = \oint \vec{A} \cdot d\vec{l}$$
$$= 2\pi s A_\phi$$
$$= \frac{2\pi s \mu_0 |M_0| \pi b^2 L}{4\pi s^2}$$
$$= \frac{\mu_0 |M_0| \pi b^2 L}{2s}$$
$$= \Phi_B \frac{L}{2s} .$$

When $\Phi_s = 0.01 \Phi_B$, 99% of the flux Φ_B will have been returned through the area bounded by the circle of radius s. Therefore

$$0.01 \Phi_B = \Phi_s$$
$$= \Phi_B \frac{L}{2s}$$
$$s = 50L .$$

Problem 3. (40 points)
Consider a "flux tube" consisting of a thin cylindrical rod of radius a, composed of linear material with constant magnetic permeability $\mu/\mu_0 \equiv \mu_r \gg 1$. The rod is finely wound with n turns/m of ideally conducting wire. It is bent into a closed circle of radius $b \gg a$ to form a thin toroid that is centered at the origin with $\hat{z}$ as its axis of rotational symmetry.

(a) (15 points) What magnitude $|\mathcal{E}|$ of EMF must be applied to the wire circuit in order to cause the current $I(t)$ flowing in the wire to increase at the rate $dI/dt = \alpha$?

Solution:
The |EMF| $|\mathcal{E}|$ is $L|dI/dt| = L|\alpha|$, where L is the self-inductance of the thin toroid:

$$L = \frac{\Phi}{I}$$
$$= \pi a^2 (2\pi bn) \frac{B}{I}$$
$$= \pi a^2 (2\pi bn) \frac{\mu n I}{I}$$
$$= 2\pi^2 a^2 b \mu n^2$$
$$|\mathcal{E}| = 2\pi^2 a^2 b \mu n^2 |\alpha| .$$

(Here we have approximated the toroidal field as that of a long solenoid as in Problem 2, taking advantage of the fact that $a \ll b$.)

(b) (10 points) Write the Biot-Savart law (relating $d\vec{B}$ to $I \, d\vec{l}$). Then, taking advantage of the duality between the equations

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \text{ and}$$
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} ,$$

write a Biot-Savart-like law, true in the limit that a is negligible, that relates $d\vec{E}$ to

$$\frac{d\Phi_B}{dt}\, \vec{dl}\,,$$

where Φ_B is the magnetic flux carried by the flux tube and $\vec{dl}$ is directed along the tube's axis.

Solution:

$$\frac{4\pi}{\mu_0}\, d\vec{B} = \frac{I\, \vec{dl} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$4\pi\, d\vec{E} = -\frac{\frac{d\Phi_B}{dt}\, \vec{dl} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}\,.$$

(**c**) (15 points) For the conditions of part (**a**), using the result of part (**b**), find the electric field that is induced at the origin.

Solution:

Since the origin is at the center of the loop, $\vec{r} = 0$. The vector $\vec{r}' = -b\hat{s}$ since it points from an element of circumference to the origin. Therefore the direction of the cross product is $\hat{\phi} \times (-\hat{s}) = +\hat{z}$. The electric field is given by

$$4\pi d\vec{E} = -\frac{d\Phi_B}{dt}\frac{b\, d\phi}{b^2}\hat{z}$$

$$4\pi \vec{E} = -\frac{d\Phi_B}{dt}\frac{2\pi}{b}\hat{z}$$

$$= -\pi a^2 \mu n \frac{dI}{dt}\frac{2\pi}{b}\hat{z}$$

$$= -\pi a^2 \mu n \alpha \frac{2\pi}{b}\hat{z}$$

$$E = -\frac{\pi a^2 \mu n \alpha}{2b}\hat{z}\,.$$

(Either sign of answer is acceptable.)